GAP THEOREM FOR SEPARATED SEQUENCES WITHOUT PAIN

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ABSTRACT. We give a simple and straightforward proof of the Gap Theorem for separated sequences by A. Poltoratski and M. Mitkovski using the Beurling–Malliavin formula for the radius of completeness.

1. Introduction and main result

For a real discrete set Λ consider the system of exponentials

$$E_{\Lambda} := \{e^{i\lambda t}\}_{\lambda \in \Lambda}.$$

The famous Beurling–Malliavin theorem gives an effective formula for the *completeness* radius R_{Λ} of E_{Λ} in terms of the so-called upper Beurling–Malliavin density $D^{BM}(\Lambda)$ (to be defined below). More precisely, put

$$R(\Lambda) = \sup\{a : E_{\Lambda} \text{ is complete in } L^{2}(-a, a)\}.$$

Then the Beurling–Malliavin theorem [1] (for detailed exposition see [2, 3]) states

Theorem 1.1.
$$R(\Lambda) = \pi D^{BM}(\Lambda)$$
.

The elegance and finality of this result impresses mathematicians over 50 years. Nevertheless, the dual concept of the lower Beurling-Malliavin density $D_{BM}(\Lambda)$ had found practical use only some years ago.

Let Λ be a separated set, i.e.

(1.1)
$$d(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

Denote by $M(\Lambda)$ the set of finite complex measures supported by Λ . The gap characteristic $G(\Lambda)$ is defined by

$$G(\Lambda) = \sup\{a : \exists \mu \in M(\Lambda) \setminus \{0\} \text{ such that } \hat{\mu}(x) = 0, x \in (-a, a)\}.$$

In 2010 M. Mitkovski and A. Poltoratski [6] proved the following result:

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Theorem 1.2. Assume $\Lambda \subset \mathbb{R}$ is a separated set. Then

$$G(\Lambda) = \pi D_{BM}(\Lambda).$$

The proof of this result in [6] uses theory of model subspaces of Hardy class H^2 , theory of Toeplitz kernels and some other tools.

The aim of our paper is to show that Theorem 1.2 can be *directly derived* from Theorem 1.1. So, instead of two difficult results in harmonic analysis essentially we have only one.

It should be noted that for non-separated sequences Λ the formula for gap characteristic was recently found by A. Poltoratski [7]. This formula is much more involved and includes the concept of energy. It is not clear (at least to the authors) whether this formula also can be directly derived from the classical Beurling-Malliavin theory.

2. Proof of Theorem 1.2

Theorem 1.2 is a straightforward consequence of Theorem 1.1 and three rather elementary results stated below.

The first result shows that the upper and lower Beurling–Malliavin densities are in a sense complementary:

Proposition 2.1. Assume $\Lambda \subset \alpha \mathbb{Z}, \alpha > 0$. Then

$$D_{BM}(\Lambda) + D^{BM}(\alpha \mathbb{Z} \setminus \Lambda) = 1/\alpha.$$

Here and below we put $\alpha \mathbb{Z} = \{\alpha n : n \in \mathbb{Z}\}.$

A similar result is true for the completeness radius and the gap characteristic:

Proposition 2.2. Assume $\Lambda \subset \alpha \mathbb{Z}, \alpha > 0$. Then

$$G(\Lambda) + R(\alpha \mathbb{Z} \setminus \Lambda) = \pi/\alpha.$$

Given a separated set Λ , we consider its perturbations:

(2.1)
$$\tilde{\Lambda} = \{\lambda + \varepsilon_{\lambda} : \lambda \in \Lambda\}.$$

The third result shows that some *positive perturbations* do not change the gap characteristic:

Proposition 2.3. Assume Λ is a separated set. For every positive number $\delta < d(\Lambda)/4$, where $d(\Lambda)$ is the separation constant in (1.1), and all numbers ε_{λ} satisfying

$$(2.2) \delta/2 < \varepsilon_{\lambda} < \delta, \quad \lambda \in \Lambda,$$

the set $\tilde{\Lambda}$ in (2.1) satisfies

$$G(\tilde{\Lambda}) = G(\Lambda).$$

Observe, that condition $\delta < d(\Lambda)/4$ implies that $\tilde{\Lambda}$ itself is a separated set.

We postpone the proofs of Propositions 2.1-2.3. Now, let us prove Theorem 1.2.

Proof. We consider two cases.

(i) Assume additionally that Λ is a subset of $\alpha \mathbb{Z}$, for some $\alpha > 0$. In view of Theorem 1.1 and Propositions 2.1 and 2.2, we have

$$G(\Lambda) = \pi/\alpha - R(\alpha \mathbb{Z} \setminus \Lambda) = \pi/\alpha - \pi D^{BM}(\alpha \mathbb{Z} \setminus \Lambda)$$
$$= \pi/\alpha - \pi(1/\alpha - D_{BM}(\Lambda)) = \pi D_{BM}(\Lambda).$$

Theorem 1.2 is proved for the subsequences of $\alpha \mathbb{Z}$.

(ii) Fix any separated set Λ and positive $\delta < d(\Lambda)/4$. Clearly, there is a set $\tilde{\Lambda}$ (2.1) satisfying (2.2) and such that $\tilde{\Lambda} \subset \alpha \mathbb{Z}$, for some sufficiently small $\alpha > 0$. By Proposition 2.3, $G(\tilde{\Lambda}) = G(\Lambda)$.

Using the definition of lower Beurling–Malliavin density (see below), one may easily check that $D_{BM}(\tilde{\Lambda}) = D_{BM}(\Lambda)$. So, by (i), we conclude that

$$G(\Lambda) = G(\tilde{\Lambda}) = \pi D_{BM}(\tilde{\Lambda}) = \pi D_{BM}(\Lambda).$$

So, we have used Theorem 1.1 for *separated sets* to deduce Theorem 1.2. We notice, that in fact these two results are equivalent. The converse implication is given by

Remark 2.4. Beurling–Malliavin's Theorem 1.1 for separated sets follows from Theorem 1.2.

To check this, one may use a similar proof where instead of Propositions 2.3 one needs

Proposition 2.5. Assume Λ is a separated set. There exists $\delta > 0$ such that for all numbers $|\varepsilon_{\lambda}| < \delta, \lambda \in \Lambda$, the set $\tilde{\Lambda}$ in (2.1) satisfies

$$R(\tilde{\Lambda}) = R(\Lambda).$$

Clearly, this result easily follows from Theorem 1.1 and the definition of D^{BM} . We remark that one may prove it by elementary means involving standard estimates of Weierstrass products.

3. Proof of Proposition 2.1

There exist at least five definitions of the upper Beurling–Malliavin density (see paper [4] which is devoted to equivalence of different definitions). We start with the most well-known:

Definition 1. We will say that the sequence $\Lambda \subset \mathbb{R}$ is strongly a-regular if its counting function n_{Λ} satisfies

$$\int_{\mathbb{R}} \frac{|n_{\Lambda}(x) - ax|}{1 + x^2} dx < \infty.$$

Definition 2. The upper Beurling–Malliavin density $D^{BM}(\Lambda)$ is the infimum of numbers a such that the function $n_{\Lambda \cup \Lambda'}$ is strongly a-regular for some $\Lambda' \subset \mathbb{R}$.

This definition goes back to J.-P.Kahane. The original definition given by Beurling and Malliavin used the notion of *short system of intervals*, see [4, p. 397–398]. We need one more equivalent definition which was found by R. Redheffer, see [8, 9].

Definition 3. The upper Beurling-Malliavin density $D^{BM}(\Lambda)$ is the infimum of numbers a such that there exists a sequence of distinct integers n_k such that

$$\sum_{\lambda_k \in \Lambda} \left| \frac{1}{\lambda_k} - \frac{a}{n_k} \right| < \infty.$$

Now we give a "dual" definition of the lower Beurling-Malliavin density.

Definition 4. The lower Beurling–Malliavin density $D_{BM}(\Lambda)$ is the supremum of numbers a such that the function $n_{\Lambda'}$ is strongly a-regular for some $\Lambda' \subset \Lambda$.

From the equivalence of Definitions 2 and 3 it follows that if $D^{BM}(\Lambda) = a$, then for every b > a there exists $\Lambda_0 \subset b^{-1}\mathbb{Z}$ such that $n_{\Lambda} - n_{\Lambda_0} \in L^1((1+x^2)^{-1}dx)$. Hence, for every b > a the sequence Λ' in Definition 2 can be taken as a subset of the arithmetic progression $b^{-1}\mathbb{Z}$.

Let us now prove Proposition 2.1. For simplicity, using re-scaling, we may assume that $\alpha = 1$.

Proof. Set $\Gamma := \mathbb{Z} \setminus \Lambda$. First of all we will show that if $D^{BM}(\Gamma) = a$, then for any b > a we can choose $\Gamma' \subset \Lambda$ such that $n_{\Gamma \cup \Gamma'}$ is strongly b-regular. Indeed, let as above $\Gamma_0 = \{b^{-1}n_k\} \subset b^{-1}\mathbb{Z}$ (where n_k are distinct integers as in Definition 3) and $n_{\Gamma} - n_{\Gamma_0} \in L^1((1+x^2)^{-1}dx)$. Put $\Gamma_1 = b^{-1}\mathbb{Z} \setminus \Gamma_0$. We have that $\Gamma \cup \Gamma_1$ is strongly b-regular. It would be natural to put $\Gamma' = \{[\gamma] : \gamma \in \Gamma_1\}$. However with this definition it is possible that $\Gamma' \cap \Gamma \neq \emptyset$. To avoid this we define

$$\Gamma'_{ex} = \{ \gamma_k \in \Gamma : \gamma_k \in [\Gamma_1] \}$$

and shift the points from Γ'_{ex} in the following way:

$$\Gamma' = ([\Gamma_1] \setminus \Gamma'_{ex}) \cup \{[b^{-1}n_k] : \gamma_k \in \Gamma'_{ex}\}.$$

Using again the fact that $n_{\Gamma} - n_{\Gamma_0} \in L^1((1+x^2)^{-1}dx)$ and that $[b^{-1}n_k] \notin [\Gamma_1]$, $\gamma_k \in \Gamma'_{ex}$ we get that $n_{\Gamma \cup \Gamma'}$ is strongly b-regular.

Now suppose that $n_{\Gamma \cup \Gamma'}$ is strongly a-regular for some $\Gamma' \subset \Lambda$. Then $n_{\mathbb{Z}\setminus(\Gamma \cup \Gamma')}$ is strongly (1-a)-regular. Since $\mathbb{Z}\setminus(\Gamma \cup \Gamma') \subset \Lambda$, we have $D_{BM}(\Lambda) \geq 1-a$ whence $D^{BM}(\Gamma)+D_{BM}(\Lambda) \geq 1$.

On the other hand, if $n_{\Lambda''}$ is strongly (1-a)-regular for some $\Lambda'' \subset \Lambda$, then $n_{\mathbb{Z}\backslash \Lambda''}$ is strongly a-regular and $\Gamma \subset \mathbb{Z} \setminus \Lambda''$. So, $D^{BM}(\Gamma) + D_{BM}(\Lambda) \leq 1$.

4. Proof of Proposition 2.2

Proof. Again, we may assume that $\alpha = 1$ and put $\Gamma := \mathbb{Z} \setminus \Lambda$. It is clear that $R(\Gamma), G(\Lambda) \leq 2\pi$.

If the system $E_{\Gamma} := \{e^{i\gamma t}\}_{\gamma \in \Gamma}$ is not complete in $L^2(0,2a)$, $0 < a < \pi$, then there exists a non-trivial function $f \in L^2(\mathbb{R})$ which vanishes outside (0,2a) and $f \perp E_{\Gamma}$. Take any small positive number ϵ and consider the convolution g = f * h, where h is a smooth function supported by $[0, \varepsilon]$. Then g is smooth, vanishes outside $(0, 2a + \varepsilon)$ and is orthogonal to E_{Γ} . Since $\{e^{int}\}_{n \in \mathbb{Z}}$ is an orthogonal basis in $L^2(0, 2\pi)$ we obtain

$$g(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx} = \sum_{n \in \mathbb{Z} \setminus \Lambda} a_n e^{inx}, \quad \{a_n\} \in \ell^1.$$

So, the measure

$$\mu := \sum_{n \in \Gamma} a_n \delta_n$$

belongs to $M(\Lambda)$ and has a spectral gap of length at least $2\pi - 2a - \varepsilon$. Since ε is arbitrary, we conclude that $R(\Gamma) + G(\Lambda) \ge \pi$.

Now, suppose that there exists a non trivial measure

$$\mu := \sum_{n \in \Lambda} a_n \delta_n \in M(\Lambda)$$

with a spectral gap of size 2a. Without loss of generality we can assume that $\hat{\mu} \equiv 0$ on (0,2a). Put $g(x) = \hat{\mu}|_{(0,2\pi)}$. We have $g \in L^2(0,2\pi)$ and $g \perp E_{\Gamma}$. Hence, $R(\Gamma) \leq \pi - a$. So, $R(\Gamma) + G(\Lambda) \leq \pi$ and Proposition 2.2 is proved.

5. Proof of Proposition 2.3

We will use the following well-known fact (see e.g. [6, Lemma 2]). For the sake of completeness we give its proof here.

Lemma 5.1. Let $\mu \in M(\mathbb{R})$. Then the Fourier transform of μ vanishes on [-a,a] if and only if

(5.1)
$$\lim_{y \to \pm \infty} e^{by} \int_{\mathbb{R}} \frac{d\mu(t)}{iy - t} = 0,$$

for every $b \in (-a, a)$.

Proof. Let μ be such that $\int_{\mathbb{R}} e^{ibt} d\mu(t) = 0$, $|b| \leq a$. Then, for any $z \in \mathbb{C}$,

$$\int_{\mathbb{R}} \frac{e^{ibt} - e^{ibz}}{t - z} d\mu(t) = ie^{ibz} \int_{\mathbb{R}} \int_{0}^{b} e^{iu(t - z)} du \, d\mu(t) = 0.$$

Hence,

(5.2)
$$\lim_{y \to \pm \infty} iy e^{by} \int_{\mathbb{R}} \frac{d\mu(t)}{iy - t} = \lim_{y \to \pm \infty} iy \int_{\mathbb{R}} \frac{e^{ibt}}{iy - t} d\mu(t) = \int_{\mathbb{R}} e^{ibt} d\mu(t) = 0.$$

Conversely, for any $b \in (-a, a)$ put

$$H(z) := \int_{\mathbb{R}} \frac{e^{ibt} - e^{ibz}}{t - z} d\mu(t).$$

Clearly H is an entire function of Cartwright class (which means that its logarithmic integral converges, see [5], Lec.16). On the other hand, by (5.2) we have $\lim_{|y|\to\infty} |H(iy)| = 0$. Hence, $H(iy) \equiv 0$ and the statement follows from (5.2).

We will also need an elementary lemma:

Lemma 5.2. Let Λ be a separated set. Then

- (i) $G(\Lambda) = G(\Lambda x)$, for every $x \in \mathbb{R}$, where $\Lambda x := \{\lambda x : \lambda \in \Lambda\}$;
- (ii) $G(\Lambda \cup \{\lambda'\}) = G(\Lambda)$, for every $\lambda' \notin \Lambda$;

(iii) if $G(\Lambda) > 0$, then for every positive $a < G(\Lambda)$ there is a measure

$$\mu = \sum_{\lambda \in \Lambda} c_{\lambda} \delta_{\lambda}$$

with spectral gap [-a, a] and such that

$$|c_{\lambda}| = O(|\lambda|^{-2}), \qquad |\lambda| \to \infty.$$

Let us, for example, check (iii). Take a positive ε satisfying $a + \varepsilon < G(\Lambda)$, and choose any measure ν with spectral gap on $[-a - \epsilon, a + \epsilon]$. Then put $\mu = h\nu$, where h is a fast decreasing function whose spectrum lies on $[-\epsilon, \epsilon]$.

Now, we prove Proposition 2.3.

Proof. In the proof below we will assume that $G(\Lambda) > 0$, and show that $G(\tilde{\Lambda}) \geq G(\Lambda)$ for every $\tilde{\Lambda}$ satisfying the assumptions of Proposition 2.3. The same proof works as well in the opposite direction: If $G(\tilde{\Lambda}) > 0$ then $G(\Lambda) \geq G(\tilde{\Lambda})$. It will follow that $G(\tilde{\Lambda}) = G(\Lambda)$. It also shows that $G(\tilde{\Lambda}) = 0$ if $G(\Lambda) = 0$.

The proof will consist of several steps.

1. We may write

$$\Lambda = \{\lambda_j : j \in \mathbb{Z}\}, \qquad \tilde{\Lambda} = \{\tilde{\lambda_j} : j \in \mathbb{Z}\},$$

where

$$\delta/2 < \tilde{\lambda_i} - \lambda_i < \delta, \qquad j \in \mathbb{Z}.$$

We may also assume that $0 \notin \tilde{\Lambda} \cup \Lambda$.

2. Consider the meromorphic function

$$\varphi(z) := -\prod_{j \in \mathbb{Z}} \frac{1 - z/\lambda_j}{1 - z/\tilde{\lambda_j}}.$$

One may check that the product converges (see, for example, [5], p. 220).

Since

$$\arg \frac{1 - z/\lambda_j}{1 - z/\tilde{\lambda}_j} = \arg(z - \lambda_j) - \arg(z - \tilde{\lambda}_j),$$

and since Λ and $\tilde{\Lambda}$ are interlacing, one may see that $\Im \varphi(z) > 0$ whenever $\Im z > 0$. Hence (see [5], p. 220, 221), φ admits a representation

$$\varphi(z) = b_1 z + b_2 + \sum_{\tilde{\lambda}_k \in \tilde{\Lambda}} c_k \left(\frac{1}{\tilde{\lambda}_k - z} - \frac{1}{\tilde{\lambda}_k} \right),$$

where $b_1 \geq 0, c_k > 0, b_2 \in \mathbb{R}$ and

$$(5.3) \sum_{k} \frac{c_k}{\tilde{\lambda}_k^2} < \infty.$$

Clearly, we have

$$(5.4) |\varphi(iy)| = O(|y|), |y| \to \infty.$$

3. Fix a positive number $a < G(\Lambda)$, and take a measure

$$\mu = \sum_{j \in \mathbb{Z}} d_j \delta_{\lambda_j}$$

which has a spectral gap on [-a, a] and whose coefficients satisfy

(5.5)
$$|d_j| = o(|j|^{-2}), \quad |j| \to \infty.$$

Then fix two points $x_1, x_2 \notin \Lambda \cup \tilde{\Lambda}$ and consider the meromorphic function

$$\psi(z) := \frac{\varphi(z)}{(z - x_1)(z - x_2)} \sum_{i \in \mathbb{Z}} \frac{d_i}{z - \lambda_i}.$$

It is easy to check that

$$\psi(z) = \sum_{k \in \mathbb{Z}} \frac{e_k}{z - \tilde{\lambda}_k} + \sum_{j=1}^2 \frac{f_j}{z - x_j},$$

where

$$e_k = \lim_{z \to \tilde{\lambda}_k} (z - \tilde{\lambda}_k) \psi(z) = \frac{c_k}{(\tilde{\lambda} - x_1)(\tilde{\lambda} - x_2)} \sum_{j \in \mathbb{Z}} \frac{d_j}{\lambda_j - \tilde{\lambda}_k}.$$

Since $|\lambda_j - \tilde{\lambda_k}| > \delta/2$, by (5.3) and (5.5) we see that $\{e_k : k \in \mathbb{Z}\} \in l^1$.

4. By Lemma 5.1, we have

$$e^{b|y|} \sum_{j \in \mathbb{Z}} \frac{d_j}{iy - \lambda_j} \to 0, \quad |y| \to \infty, \text{ for every } 0 < b < a.$$

So, by (5.4), the same estimate holds for the function

$$\sum_{k \in \mathbb{Z}} \frac{e_k}{iy - \tilde{\lambda}_k} + \sum_{j=1}^2 \frac{f_j}{iy - x_j}.$$

This shows that $G(\tilde{\Lambda} \cup \{x_1, x_2\}) \geq a$. Since this is true for every $a < G(\Lambda)$, by Lemma 5.2 (ii), we conclude that $G(\tilde{\Lambda}) \geq G(\Lambda)$.

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